Admissible initial growth for diffusion equations with weakly superlinear absorption

Andrey Shishkov

Institute of Applied Mathematics and Mechanics of NAS of Ukraine,
R. Luxemburg str. 74, 83114 Donetsk, Ukraine

Laurent Véron

Laboratoire de Mathématiques et Physique Théorique, CNRS UMR 6083,
Université François-Rabelais, 37200 Tours, France

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Abstract We study the admissible growth at infinity of initial data of positive solutions of $\partial_t u - \Delta u + f(u) = 0$ in $\mathbb{R}_+ \times \mathbb{R}^N$ when $f(u)$ is a continuous function, *mildly* superlinear at infinity, the model case being $f(u) = u \ln^\alpha(1+u)$ with $1 < \alpha < 2$. We prove in particular that if the growth of the initial data at infinity is too strong, there is no more diffusion and the corresponding solution satisfies the ODE problem $\partial_t \phi + f(\phi) = 0$ on $\mathbb{R}_+$ with $\phi(0) = \infty$.

Contents

1 Introduction and formulation of the results

Let h be a continuous nondecreasing function defined on $\mathbb{R}_+$ and vanishing only at 0. It is well known that for any continuous and bounded function g belonging to $C_b^+(\mathbb{R}^N)$, the cone of bounded nonnegative continuous functions on $\mathbb{R}^N$, there exists a unique weak solution $u := u_g \in C_b^+(\mathbb{R}_+ \times \mathbb{R}^N)$ of

$$\begin{aligned} \partial_t u - \Delta u + uh(u) &= 0 && \text{in } Q_{\mathbb{R}^N}^\infty := \mathbb{R}_+ \times \mathbb{R}^N, \\ \lim_{t\to 0} u(t,.) &= g && \text{locally uniformly in } \mathbb{R}^N. \end{aligned} \tag{1.1}$$

Furthermore, the solution u satisfies

$$0 \le u(x,t) \le \Phi_{\|g\|_{L^\infty}}(t) \qquad \forall (t,x) \in Q_{\mathbb{R}^N}^\infty, \tag{1.2}$$

where Φ_a is the solution of the following Cauchy problem:

$$\begin{aligned} \Phi_t + \Phi h(\Phi) &= 0 \qquad \text{on } \mathbb{R}_+, \\ \Phi(0) &= a. \end{aligned} \tag{1.3}$$

Since $h(0) = 0$ it follows easily that

$$\Phi_a(t) \geqslant 0 \quad \forall t > 0, \forall a \geqslant 0.$$

Moreover the family $\{\Phi_a(t)\}$ is monotonically increasing with respect to the parameter a, and the condition:

$$\int_c^\infty \frac{ds}{sh(s)} < \infty, \quad c = \text{const} > 0, \tag{1.4}$$

is equivalent to the existence of a solution $\Phi_\infty(t)$ of equation in (1.3) such that $\Phi_\infty(t) < \infty$, $\forall t > 0$ and with infinite initial data, i. e.

$$\lim_{t\to 0} \Phi_\infty(t) = \infty.$$

Now we come to our main subject, to study problem (1.1) with an initial data $g(x)$ unbounded and tending to infinity at infinity. It is clear that the character of growth of $h(s)$ at infinity defines the class of initial functions g *of solvability* of problem under consideration. For example, if $h(s)$ is bounded, then the corresponding class of solvability is the Tikhonov class [7] $\{g : g(x) \leqslant c\exp(c_1|x|^2), c, c_1 = \text{const} < \infty\}$. When $h(s)$ tends to infinity at infinity, the class of admissible initial data is larger that the Tikhonov class. If $h(s)$ increases at infinity fast enough in the sense that condition (1.4) is satisfied, then problem (1.1) is solvable for any nonnegative continuous function g in the sense that there exists a *prospective minimal solution* $\underline{u}_g$ which is the limit when $n \to \infty$ of the solutions u_{g_n} of

$$\begin{aligned} \partial_t u - \Delta u + uh(u) &= 0 && \text{in } Q_{\mathbb{R}^N}^\infty \\ \lim_{t\to 0} u(t,.) &= g\chi_{B_n} && \text{in } L^1(\mathbb{R}^N), \end{aligned} \tag{1.5}$$

where B_n denotes the open ball of radius n and χ_A is characteristic function of the set A, and there holds

$$0 \le \underline{u}_g \le \Phi_\infty. \tag{1.6}$$

But the main question is to know whether the prospective minimal solution is truly a solution with initial data $g(.)$. If it is the case we say that the prospective minimal solution is the minimal solution. Another assumption on h which plays a fundamental role in the study is the so-called Keller-Osserman condition (see [1], [6]),

$$\int_a^\infty \frac{ds}{\sqrt{H(s)}} < \infty \text{ for some } a > 0, \text{ where } H(t) = \int_0^s th(t)dt. \tag{1.7}$$

When this condition is satisfied, for any $R > 0$ there exist solutions of

$$\begin{aligned} -\Delta u + uh(u) = 0 \qquad &\text{in } B_R \\ \lim_{|x|\to R} u(x) = \infty. & \end{aligned} \tag{1.8}$$

This condition gives rise to a localization phenomenon thanks to which we prove an existence and uniqueness result of the solution with initial data g.

Theorem A *Assume* $r \mapsto rh(r)$ *is convex and satisfies* (1.7). *Then for any* $g \in C^+(\mathbb{R}^N)$, $\underline{u}_g$ *is the minimal solution with initial data* g. *Furthermore it is the unique nonnegative solution of problem* (1.1).

In the class of functions $h(s)$ of the form $h(s) = s\ln^\alpha(1+s)$, condition (1.4) is equivalent to $\alpha > 1$, but condition (1.7) is equivalent to $\alpha > 2$. In general it is easy to show that condition (1.7) is stronger than condition (1.4) .

When $h(s)$ is a power function, the class of existence and uniqueness is much larger than the class of Th. A. A complete description of this existence and uniqueness class is based upon the notion of initial trace which has been thoroughly investigated by Marcus and Véron [3], [4] and Gkikas and Véron [2].

When

$$\int_a^\infty \frac{ds}{\sqrt{H(s)}} = \infty \qquad \forall a > 0, \tag{1.9}$$

uniqueness may not hold in the class of unbounded solutions. If, for any $b > 0$, V_b denotes the maximal solution of the following Cauchy problem

$$\begin{aligned} V_{rr} + \frac{N-1}{r} V_r - Vh(V) = 0 \qquad &\text{on } (0, R_b) \\ V(0) = b & \\ V_r(0) = 0, & \end{aligned} \tag{1.10}$$

then $R_b = \infty$. Actually, multiplying (1.10) by V_r, we get easily

$$2^{-1}\frac{d}{dr}|V_r|^2 = \frac{d}{dr}H(V) - \frac{N-1}{r}|V_r|^2 \leqslant \frac{d}{dr}H(V).$$

Since $V_r(0) = 0$, we derive

$$V_r(r) \leqslant \sqrt{2}\sqrt{H(V(r))} \qquad \forall r > 0.$$

Integrating this last inequality we obtain the *a priori* estimate

$$V(R)=V_b(R)\leqslant \bar{V}_b(R) \qquad \forall R>0,$$

where the function $\bar{V}_b(R)$ is defined by the identity:

$$F_b(\bar{V}_b(R))=\sqrt{2}R \qquad \forall R>0, \text{ where } F_b(v):=\int_b^v \frac{ds}{\sqrt{H(s)}},$$

(see e.g. [8] also). Moreover it is easy to see that for arbitrary $a>b>0$, $V_a(r)\geqslant V_b(r)$ $\forall r>0$. Actually, due to the monotonicity of h, there holds

$$V_{a_{rr}}(0)=\frac{1}{N}V_a(0)h(V_a(0))=\frac{1}{N}ah(a)>\frac{1}{N}bh(b)=V_{b_{rr}}(0),$$

from (1.10). Since $V_{a_r}(0)=V_{b_r}(0)=0$ it follows from this last inequality that the function $r\mapsto W(r)=V_a(r)-V_b(r)$ is increasing near $r=0$; it remains increasing on whole $\mathbb{R}_+$, since, if we assume that there exists $r_0>0$ where W reaches a local maximum, then $W_r(r_0)=0, W_{rr}(r_0)\leq 0$, but from equation (1.10) we have:

$$W_{rr}(r_0)=V_a(r_0)h(V_a(r_0))-V_b(r_0)h(V_b(r_0))>0,$$

which is a contradiction. Furthermore, Nguyen Phuoc and Véron proved in [5] that if g satisfies

$$V_c(|x|)\leq g(x)\leq V_b(|x|) \qquad \forall x\in\mathbb{R}^N, \tag{1.11}$$

for some $b>c>0$, then there exists at least two different solutions of (1.1) defined in $Q^\infty_{\mathbb{R}^N}$: the minimal one $\underline{u}_g$ which satisfies

$$\underline{u}_g(x,t)\leq \Phi_\infty(t) \qquad \forall (x,t)\in Q^\infty_{\mathbb{R}^N}, \tag{1.12}$$

and another one u_g such that

$$V_c(|x|)\leq u_g(x,t)\leq V_b(|x|) \qquad \forall (x,t)\in Q^\infty_{\mathbb{R}^N}. \tag{1.13}$$

It is not clear wether there exists a maximal solution or not. However, if g satisfies (1.11), then there exists a minimal solution $\underline{u}_{g,c,b}$ and a maximal one $\overline{u}_{g,c,b}$ in the class $\mathcal{E}_{c,b}(g)$ of solutions of problem (1.1), satisfying inequalities (1.13). These two solutions can be constructed by the following approximate scheme: we define the sequence $\{\underline{u}_n\}$ of solutions of the Cauchy-Dirichlet problem

$$\begin{aligned} \partial_t u-\Delta u+uh(u)&=0 && \text{in } Q^\infty_{B_n}:=\mathbb{R}_+\times B_n\\ u(t,x)&=V_c(n) && \text{in } \partial_\ell Q^\infty_{B_n}:=\mathbb{R}_+\times\partial B_n\\ u(0,.)&=g && \text{in } B_n; \end{aligned} \tag{1.14}$$

then it is easy to check using comparison principle that the sequence $\{\underline{u}_n\}$ is increasing and converges to $\underline{u}_{g,c,b}$. Similarly, the sequence $\{\overline{u}_n\}$ of solutions of the same equation in

$Q^\infty_{B_n}$ with the same initial data and boundary value $V_b(n)$ is decreasing and converges to $\overline{u}_{g,c,b}$.

In this paper we consider the case where the initial data g grows at infinity faster than any function V_b with arbitrary $b < \infty$. Our aim is to describe analogs of the "maximal" solution u_g from (1.13) and prospective minimal solution $\{\underline{u}_g\}$ from (1.5), (1.6). For any $a > 0$ we denote by $u := u_{a,n}$ the solution of

$$\begin{aligned} \partial_t u - \Delta u + uh(u) = 0 \qquad & \text{in } Q^\infty_{B_n} := \mathbb{R}_+ \times B_n \\ u(t,x) = V_a(n) \qquad & \text{in } \partial_\ell Q^\infty_{B_n} := \mathbb{R}_+ \times \partial B_n \\ u(0,.) = \min\{V_a, g\} \qquad & \text{in } B_n, \end{aligned} \tag{1.15}$$

Due to the comparison principle it is clear that $u_{a,n} \leqslant V_a$ in $Q^\infty_{B_n}$. The next result highlights a phenomenon of *instantaneous blow-up* of the maximal solution if the initial data grows too fast at infinity.

Theorem B *Assume $r \mapsto rh(r)$ is convex and satisfies* (1.4) *and* (1.9)*. If $g \in C^+(\mathbb{R}^N)$, satisfies*

$$\lim_{|x|\to\infty} \frac{g(x)}{V_a(|x|)} = \infty \qquad \forall a > 0, \tag{1.16}$$

then for arbitrary $m \in \mathbb{N}$ the sequence $\{u_{a,n}\}_{n>m}$ decreases and converges in $Q^\infty_{B_m}$ to a function u_a which is solution of (1.1) *with initial data* $\min\{V_a, g\}$*. Furthermore $u_a(t,x) \to \infty$ for any $(t,x) \in Q^\infty_{\mathbb{R}^N}$ as $a \to \infty$. Thus, the function identically equal to ∞ in $Q^\infty_{\mathbb{R}^N}$ can be considered as the "maximal" solution of problem* (1.1) *in the case of* (1.16).

Let us remark that in subsection 3.1 we find the asymptotic expression of the functions V_a for the model nonlinearities h, which makes the condition (1.16) more explicit.

A fundamental example of equations with nonlinearities satisfying (1.4) and (1.9) is provided by

$$\partial_t u - \Delta u + u \ln^\alpha(1+u) = 0 \qquad \text{in } Q^\infty_{\mathbb{R}^N} \tag{1.17}$$

with $1 < \alpha \leq 2$. With this specific type of nonlinearity we prove:

Theorem C *Assume $1 < \alpha < 2$ and $g \in C^+(\mathbb{R}^N)$, satisfies condition* (1.16)*, which due to Proposition 3.1 has now the following form*

$$\lim_{|x|\to\infty} g(x) \exp\left(-c_\alpha |x|^{\frac{2}{2-\alpha}}\right) = \infty, \quad c_\alpha = \left(\frac{2-\alpha}{2}\right)^{\frac{2}{2-\alpha}}.$$

Then the prospective minimal solution $\underline{u}_g$ of (1.17) *with initial data g is Φ_∞.*

Notice that the two types of generalized approximative solutions of problem (1.1), obtained in Theorems B, C, "forget" the real initial condition from (1.1): in another words, they realize infinite initial jump.

2 The maximal solution

2.1 Proof of Theorem A

The fact that $\underline{u}_g$ is a solution of (1.1), and clearly the minimal one, is more or less standard, but we recall its proof for the sake of completeness since it contains the localization principle. For $m \in \mathbb{N}^*$ let v_m be the minimal solution of

$$\begin{array}{c}-\Delta v + vh(v) = 0 \qquad \text{in } \ B_m \\ \lim_{|x| \to m} v(x) = \infty.\end{array} \tag{2.1}$$

Such a solution exists by [1] or [6] because (1.7) holds. It is nonnegative and radial as limit of the nonnegative radial functions $v_{m,k}$, $k \in \mathbb{N}^*$ which are the solutions of (2.1) with finite boundary data $v_{m,k} = k$ on ∂B_m. Moreover $v_{m,k}$, and thus v_m, is an increasing function of $|x|$. Clearly $v_m \geq 0$ and it is a stationary solution of (1.1) in $Q^\infty_{B_m}$. For $n \geq m$, let u_{g_n} be the solution of (1.5) and $\gamma_m = \max\{g(x) : |x| \leq m\}$. Then $v_m + \gamma_m$ is a super solution of (1.5) in $Q^\infty_{B_m}$ which dominates u_n on $\partial_\ell Q^\infty_{B_m} \cup \{0\} \times B_m$ Thus $v_m + \gamma_m \geq u_n$ in $Q^\infty_{B_m}$. The set of functions $\{u_n\}$ is bounded and uniformly continuous in $Q^T_{B_{m-1}}$, by standard regularity theory for parabolic equations, thus it converges uniformly therein to $\underline{u}_g$ and $\underline{u}_g\lfloor_{B_m \times \{0\}} = g$. This implies that $\underline{u}_g$ has g as initial data.

Assume now that u another solution with the same initial data g. We set $w = u - \underline{u}_g$. Since $r \mapsto rh(r)$ is convex and $u - \underline{u}_g$ is positive,

$$uh(u) \geq \underline{u}_g h(\underline{u}_g) + (u - \underline{u}_g)h(u - \underline{u}_g).$$

Therefore w is a subsolution of problem (1.1), and $w(t,x) \to 0$ as $t \to 0$, locally uniformly in $\mathbb{R}^N$. By the comparison principle

$$w(t,x) \leq v_n(x) \qquad \text{in } \ Q^\infty_{B_n},$$

where v_n satisfies (2.1) in B_n. Furthermore $n \mapsto v_n$ is decreasing with limit v_∞ as $n \to \infty$. The function v_∞ verifies

$$-\Delta v + vh(v) = 0 \qquad \text{in } \ \mathbb{R}^N.$$

Furthermore it is nonnegative, radial and nondecreasing with respect to $|x|$. In order to prove that $v = 0$, we return to v_n which satisfies

$$v_{n\,r} = r^{1-N} \int_0^r s^{N-1} v_n(s) h(v_n(s)) ds \leq v_n(r) h(v_n(r)) r^{1-N} \int_0^r s^{N-1} ds = \frac{r}{N} v_n(r) h(v_n(r)).$$

Thus

$$-v_{n\,rr} + v_n(r)h(v_n(r)) = \frac{N-1}{r} v_{n\,r} \leq \left(1 - \frac{1}{N}\right) v_n(r) h(v_n(r))$$

which implies

$$-v_{n\,rr} + \frac{1}{N} v_n(r) h(v_n(r)) \leq 0.$$

Integrating twice yields

$$\int_{v_n(r)}^{\infty} \frac{ds}{\sqrt{H(t)}} \geq \sqrt{\frac{2}{N}}(n-r), \tag{2.2}$$

where H has been defined in (1.7). If we had $v_\infty(r) > 0$ for some $r > 0$, it would imply

$$\infty > \int_{v_\infty(r)}^{\infty} \frac{ds}{\sqrt{2H(t)}} \geq \infty,$$

a contradiction. Thus $v_\infty(r) = 0$ and $w(t,x) = 0$. $\square$

2.2 Proof of Theorem B

We recall that (1.16) holds and that $u_{a,n}$ denotes the solution of (1.15). Since $V_a\lfloor_{Q^\infty_{B_n}}$ is the solution of the Cauchy-Dirichlet problem

$$\begin{aligned}
\partial_t u - \Delta u + uh(u) = 0 \qquad & \text{in } Q^\infty_{B_n} := \mathbb{R}_+ \times B_n \\
u(t,x) = V_a(n) \qquad & \text{in } \partial_\ell Q^\infty_{B_n} := \mathbb{R}_+ \times \partial B_n \\
u(0,.) = V_a \qquad & \text{in } B_n,
\end{aligned} \tag{2.3}$$

it is larger than $u_{a,n}$. Thus $u_{a,n+1}\lfloor_{\partial_\ell Q^\infty_{B_n}} \leq u_{a,n}\lfloor_{\partial_\ell Q^\infty_{B_n}} = V_a$. Since $u_{a,n}(0,.) = u_{a,n+1}\lfloor_{B_n}(0,.)$ it follows that $u_{a,n+1}\lfloor_{Q^\infty_{B_n}} \leq u_{a,n}$. Then $\{u_{a,n}\}$ is a decreasing sequence, and its limit u_a is a solution of (1.1), which the first claim. By the same argument, $u_{a,n} \leq u_{b,n+1}\lfloor_{Q^\infty_{B_n}}$ in $Q^\infty_{B_n}$ for $b > a$. Hence $u_a \leq u_b$. We introduce the sequence $\{r_a\} : r_a \to \infty$ as $a \to \infty$ defined by:

$$r_a = \inf\{r > 0 : g(x) \geqslant V_a(x) \quad \forall\, |x| \geqslant r\}, \tag{2.4}$$

and, for $n \geq r_a$, we set $w_{a,n} = V_a - u_{a,n}$. By convexity $w_{a,n}$ satisfies

$$\begin{aligned}
\partial_t w_{a,n} - \Delta w_{a,n} + w_{a,n} h(w_{a,n}) \leq 0 \qquad & \text{in } Q^\infty_{B_n} := \mathbb{R}_+ \times B_n, \\
w_{a,n}(t,x) = 0 \qquad & \text{in } \partial_\ell Q^\infty_{B_n} := \mathbb{R}_+ \times \partial B_n, \\
w_{a,n}(0,x) = (V_a - g)_+ \qquad & \text{in } B_n.
\end{aligned} \tag{2.5}$$

Therefore

$$w_{a,n}(t,x) < \Phi_\infty(t) \qquad \text{in } Q^\infty_{B_n}, \tag{2.6}$$

where Φ_∞ is defined in (1.3) with $a = \infty$. Actually,

$$\int_{\Phi_\infty(t)}^{\infty} \frac{ds}{sh(s)} = t. \tag{2.7}$$

Notice also that the sequence $\{w_{a,n}\}$ is increasing and it converges, as $n \to \infty$, to $w_a = V_a - u_a$, which is dominated by Φ_∞ Thus

$$u_a(t,x) \geq V_a(x) - \Phi_\infty(t) \geq a - \Phi_\infty(t) \qquad \text{in } Q^\infty_{\mathbb{R}^N}. \tag{2.8}$$

Letting $a \to \infty$ implies the claim. $\square$

3 The prospective minimal solution

In this section we consider equation (1.17) with $1 < \alpha < 2$.

3.1 The stationary problem

Proposition 3.1 *Assume* $1 < \alpha < 2$, $a > 0$ *and* V_a *is the solution of*

$$\begin{aligned} V_{rr} + \frac{N-1}{r} V_r - V \ln^{\alpha}(V+1) = 0 \qquad &\text{in } \mathbb{R}_+ \\ V_r(0) = 0 \\ V(0) = a. \end{aligned} \tag{3.1}$$

Then

$$V_a(r) = e^{c_\alpha r^{\frac{2}{2-\alpha}} + O(1)} \qquad \text{as } r \to \infty, \tag{3.2}$$

where $c_\alpha = \left(\frac{2-\alpha}{2}\right)^{\frac{2}{2-\alpha}}$.

Proof. We write $W = \ln(V+1)$. Since V is increasing $W > 0, W_r \geq 0$ and

$$W_{rr} + W_r^2 + \frac{N-1}{r} W_r - (1 - e^{-W}) W^{\alpha} = 0 \qquad \text{in } \mathbb{R}_+. \tag{3.3}$$

Thus

$$W_{rr} + W_r^2 - (1 - e^{-W}) W^{\alpha} \leq 0.$$

If we set $\rho = W$ and $p(\rho) = W_r(r)$, then $\rho \in [a, \infty)$ and

$$pp' + p^2 - (1 - e^{-\rho}) \rho^{\alpha} \leq 0.$$

This is a linear differential inequality in the unknown p^2. Integrating yields

$$p^2(\rho) \leq 2e^{-2\rho} \int_a^{\rho} (e^{2s} - e^s) s^{\alpha} ds = \rho^{\alpha} + O(1). \tag{3.4}$$

Thus $W_r(r) \leq W^{\frac{\alpha}{2}}(r) + O(1)$ as $r \to \infty$ which implies

$$W(r) \leq c_\alpha r^{\frac{2}{2-\alpha}} + O(1) \qquad \text{as } r \to \infty. \tag{3.5}$$

Due to (3.5) relation (3.4) yields also the following inequality

$$0 < W_r \leq c_\alpha^{\frac{\alpha}{2}} r^{\frac{\alpha}{2-\alpha}} (1 + o(1)).$$

Since $W(r) \to \infty$ as $r \to \infty$, it follows from (3.3) and (3.4) that for any $\epsilon > 0$ there exists $r_\epsilon > 0$ such that

$$W_{rr} + W_r^2 \geq (1 - \epsilon) W^{\alpha} \qquad \text{on } [r_\epsilon, \infty).$$

Integrating this ordinary differential inequality we get

$$W(r) \geq (1-\epsilon))c_\alpha r^{\frac{2}{2-\alpha}}(1+o(1)) \qquad \text{as } r \to \infty. \tag{3.6}$$

Since ϵ is arbitrary, we derive

$$W(r) = c_\alpha r^{\frac{2}{2-\alpha}}(1+o(1)) \qquad \text{as } r \to \infty. \tag{3.7}$$

From the above estimates, we can improve (3.6). Using (3.4) and (3.7) we deduce from (3.3):

$$pp' + p^2 = (1-e^{-\rho})\rho^\alpha - \frac{N-1}{r}W_r \geq (1-e^{-\rho})\rho^\alpha - c\rho^{\alpha-1},$$

from which it follows easily

$$p^2(\rho) \geq 2e^{-2\rho}\int_a^\rho e^{2s}(s^\alpha - c's^{\alpha-1})ds = \rho^\alpha + O(1), \tag{3.8}$$

by l'Hospital rule. Combined with (3.7) and (3.5), it implies

$$W(r) = c_\alpha r^{\frac{2}{2-\alpha}} + O(1) \qquad \text{as } r \to \infty. \tag{3.9}$$

Returning to V_a, we derive

$$V_a(r) = e^{c_\alpha r^{\frac{2}{2-\alpha}}+O(1)} \qquad \text{as } r \to \infty. \tag{3.10}$$

$\square$

Remark. If $\alpha = 2$, the same method yields

$$V_a(r) = e^{e^r + O(1)} \qquad \text{as } r \to \infty. \tag{3.11}$$

3.2 Proof of Theorem C

We recall that the prospective minimal solution $\underline{u}_g$ is the limit, when $n \to \infty$ of the (increasing) sequence of solutions $\{u_{g_{\ell_n}}\}$ of

$$\begin{aligned} \partial_t u - \Delta u + u\ln^\alpha(u+1) &= 0 && \text{in } Q^\infty_{\mathbb{R}^N} \\ u(0,.) &= g\chi_{B_{\ell_n}} && \text{in } \mathbb{R}^N, \end{aligned} \tag{3.12}$$

where $\{\ell_n\}$ is any increasing sequence converging to ∞. Furthermore, if we replace g by its maximal radial minorant defined by $\tilde{g}(r) := \min_{|x|=r} g(x)$, it satisfies also (1.16). Because of (1.16) there exists a sequence $\{r_n\}$ tending to infinity such that

$$r_n = \inf\{r > 0 : \tilde{g}(s) \geq V_n(s) \;\; \forall s \geq r\},$$

then $\tilde{g}(r_n) = V_n(r_n)$.

Step 1: Estimate from below. Put

$$g_n(|x|) = \begin{cases} \min\{\tilde{g}(r_n), \tilde{g}(|x|)\} & \text{if } |x| < r_n \\ \tilde{g}(r_n) & \text{if } |x| \geq r_n. \end{cases}$$

Let $\underline{u}_{g_n}$ be the minimal solution of

$$\begin{aligned} \partial_t u - \Delta u + u \ln^\alpha(u+1) &= 0 && \text{in } Q^\infty_{\mathbb{R}^N} \\ u(0,.) &= g_n && \text{in } \mathbb{R}^N. \end{aligned} \tag{3.13}$$

Then $\underline{u}_{g_n} \leq \Phi_\infty$. For any sequence $\{\ell_k\}$ converging to infinity and any fixed k, there exists n_k such that for $n \geq n_k$, there holds $g\chi_{B_{\ell_k}} \leq g_n$. Since the sequence $\{\underline{u}_{g_n}\}$ is increasing, its limit u_∞ is a solution of (1.3) in $Q^{\mathbb{R}^N}_\infty$ which is larger than $u_{g_{\ell_k}}$ for any ℓ_k, and therefore larger also than $\underline{u}_{\tilde{g}}$. However, since $g_n \leq \tilde{g}$, $u_\infty \leq \underline{u}_{\tilde{g}}$. This implies

$$u_\infty = \underline{u}_{\tilde{g}} \leqslant \Phi_\infty. \tag{3.14}$$

Next, since $\underline{u}_{g_n}(0,x) \leq g(r_n)$, it follows that $\underline{u}_{g_n}(t,x) \leq g(r_n)$. Let $\omega_n = \Phi_{g(r_n)}$, i.e. the solution of (1.3) with $a = g(r_n)$, then ω_n satisfies

$$\int_{\omega_n(t)}^{g(r_n)} \frac{ds}{sh(s)} = t,$$

and $\underline{u}_{g_n} \geq w_n$ where w_n is the minimal solution of

$$\begin{aligned} \partial_t w - \Delta w + w \ln^\alpha(\omega_n+1) &= 0 && \text{in } Q^\infty_{\mathbb{R}^N} \\ w(0,.) &= g_n && \text{in } \mathbb{R}^N. \end{aligned} \tag{3.15}$$

If we set $w_n(t,x) = e^{-\int_0^t \ln^\alpha(\omega_n(s)+1)ds} z_n(t,x)$, then

$$\begin{aligned} \partial_t z_n - \Delta z_n &= 0 && \text{in } Q^\infty_{\mathbb{R}^N} \\ z_n(0,.) &= g_n && \text{in } \mathbb{R}^N. \end{aligned} \tag{3.16}$$

Since

$$z_n(t,x) = \frac{1}{(4\pi t)^{\frac{N}{2}}} \int_{\mathbb{R}^N} e^{-\frac{|x-y|^2}{4t}} g_n(y)dy,$$

we can write $w_n(t,x) = I_n(t,x) + J_n(t,x)$ where

$$I_n(t,x) = \frac{e^{-\int_0^t \ln^\alpha(\omega_n(s)+1)ds}}{(4\pi t)^{\frac{N}{2}}} \int_{|y|\leq r_n} e^{-\frac{|x-y|^2}{4t}} g_n(y)dy, \tag{3.17}$$

and

$$J_n(t,x) = \frac{e^{-\int_0^t \ln^\alpha(\omega_n(s)+1)ds} \tilde{g}(r_n)}{(4\pi t)^{\frac{N}{2}}} \int_{|y|>r_n} e^{-\frac{|x-y|^2}{4t}} dy. \tag{3.18}$$

Clearly

$$
\begin{aligned}
J_n(t,x) &\geq \frac{e^{-\int_0^t \ln^\alpha(\omega_n(s)+1)ds}\tilde{g}(r_n)}{(4\pi t)^{\frac{N}{2}}}\int_{|y|>r_n+|x|} e^{-\frac{|y|^2}{4t}}dy \\
&\geq \frac{e^{-\int_0^t \ln^\alpha(\omega_n(s)+1)ds}\tilde{g}(r_n)}{(4\pi t)^{\frac{N}{2}}}\left(\int_{|z|>r_n+|x|} e^{-\frac{z^2}{4t}}dz\right)^N.
\end{aligned} \tag{3.19}
$$

This integral term can be estimated by introducing Gauss error function

$$
\operatorname{ercf}(x) = \frac{2}{\sqrt{\pi}}\int_x^\infty e^{-z^2}dz. \tag{3.20}
$$

In dimension N, it implies easily

$$
J_n(t,x) \geq e^{-\int_0^t \ln^\alpha(\omega_n(s)+1)ds}\tilde{g}(r_n)\left(\operatorname{ercf}\left(\frac{r_n+|x|}{2\sqrt{t}}\right)\right)^N. \tag{3.21}
$$

Since

$$
\operatorname{ercf}(x) = \frac{e^{-x^2}}{x\sqrt{t}}(1+O(x^{-2})) \quad \text{as } x\to\infty,
$$

we derive

$$
J_n(t,x) \geq \frac{\tilde{g}(r_n)}{((r_n+|x|)^2t)^{\frac{N}{2}}}e^{-\int_0^t \ln^\alpha(\omega_n(s)+1)ds-\frac{N(r_n+|x|)^2}{4t}}\left(1+O\left(\frac{t}{r_n^2}\right)\right). \tag{3.22}
$$

We write $\tilde{g}(r) = \exp(\gamma(r)) - 1$ and set

$$
A_n(t,x) = \gamma(r_n) - \int_0^t \ln^\alpha(\omega_n(s)+1)ds - \frac{N(r_n+|x|)^2}{4t} - N\ln(r_n+|x|) - \frac{N}{2}\ln t.
$$

In order to have an estimate on $\omega_n(s)$, we fix $t \leq 1$ and $\tilde{g}(r_n) \geq 1$. There exists $a_0 \geq 1$ such that

$$
\min\left\{\frac{\omega_a(t)}{\omega_a(t)+1} : 0 \leq t \leq 1,\ a \geq a_0\right\} \geq \frac{1}{2}.
$$

In such a range of a and t,

$$
\begin{aligned}
\omega' + \omega\ln^\alpha(\omega+1) &= \omega' + \frac{\omega}{\omega+1}(\omega+1)\ln^\alpha(\omega+1) \\
&\geq \omega' + \frac{1}{2}(\omega+1)\ln^\alpha(\omega+1),
\end{aligned}
$$

which yields

$$
\ln^\alpha(\omega_n(s)+1) \leq \left(\frac{2\gamma^{\alpha-1}(r_n)}{2+(\alpha-1)s\gamma^{\alpha-1}(r_n)}\right)^{\frac{\alpha}{\alpha-1}}.
$$

From this inequality, we derive

$$
\begin{aligned}
\int_0^t \ln^\alpha(\omega_n(s)+1)ds &\leq \int_0^t \left(\frac{2\gamma^{\alpha-1}(r_n)}{2+(\alpha-1)s\gamma^{\alpha-1}(r_n)}\right)^{\frac{\alpha}{\alpha-1}} ds \\
&\leq 2^{\frac{\alpha}{\alpha-1}}\gamma(r_n)\int_0^{t\gamma^{\alpha-1}(r_n)} (2+(\alpha-1)\tau)^{-\frac{\alpha}{\alpha-1}} d\tau.
\end{aligned}
$$

Therefore

$$A_n(t,x) \geq \gamma(r_n) - \frac{N(r_n+|x|)^2}{4t} - N\ln(r_n+|x|) - \frac{N}{2}\ln t \\ - 2^{\frac{\alpha}{\alpha-1}}\gamma(r_n)\int_0^{t\gamma^{\alpha-1}(r_n)} (2+(\alpha-1)\tau)^{-\frac{\alpha}{\alpha-1}}\,d\tau. \tag{3.23}$$

Step 2: The maximal admissible growth. We claim that

$$\liminf_{|x|\to\infty} |x|^{-\frac{2}{2-\alpha}} \ln \tilde{g}(|x|) > N^{\frac{1}{2-\alpha}} \Longrightarrow \lim_{n\to\infty} \underline{u}_{g_n}(t,x) = \Phi_\infty(t) \qquad \forall (t,x) \in Q^\infty_{\mathbb{R}^N}. \tag{3.24}$$

By replacing $\tau \mapsto (2+(\alpha-1)\tau)^{-\frac{\alpha}{\alpha-1}}$ by its maximal value on $(0, t\gamma^{\alpha-1}(r_n))$,

$$2^{\frac{\alpha}{\alpha-1}}\gamma(r_n)\int_0^{t\gamma^{\alpha-1}(r_n)} (2+(\alpha-1)\tau)^{-\frac{\alpha}{\alpha-1}}\,d\tau \leq \gamma^\alpha(r_n)t.$$

Then

$$A_n(t,x) \geq \gamma(r_n) - \frac{N(r_n+|x|)^2}{4t} - N\ln(r_n+|x|) - \frac{N}{2}\ln t - \gamma^\alpha(r_n)t := B_n(t,x), \tag{3.25}$$

and

$$\partial_t B_n(t,x) = \frac{N(r_n+|x|)^2}{4t^2} - \frac{N}{2t} - \gamma^\alpha(r_n).$$

Thus

$$\partial_t B_n(t,x) = 0 \text{ and } t > 0 \Longleftrightarrow t := t_n = \frac{N(r_n+|x|)^2}{N + \sqrt{N^2 + 4N(r_n+|x|)^2\gamma^\alpha(r_n)}}. \tag{3.26}$$

Therefore $A_n(t_n,x)$ is bounded from below by the maximum of $B_n(t,x)$ which is achieved for $t = t_n$ and

$$B_n(t_n,x) = \gamma(r_n) - N\ln(r_n+|x|) - \frac{N + \sqrt{N^2 + 4N(r_n+|x|)^2\gamma^\alpha(r_n)}}{4} \\ - \frac{N(r_n+|x|)^2\gamma^\alpha(r_n)}{N + \sqrt{N^2 + 4N(r_n+|x|)^2\gamma^\alpha(r_n)}} - \frac{N}{2}\ln\left(\frac{N(r_n+|x|)^2}{N + \sqrt{N^2 + 4N(r_n+|x|)^2\gamma^\alpha(r_n)}}\right).$$

Since $r_n \to \infty$ as $n \to \infty$ it follows from last representation that

$$B_n(t_n,x) = r_n\gamma^{\frac{\alpha}{2}}(r_n)\left(\frac{\gamma^{1-\frac{\alpha}{2}}(r_n)}{r_n} - N^{\frac{1}{2}}(1+\nu_n(x))\right), \tag{3.27}$$

where $\nu_n(x) \to 0$ as $n \to \infty$ uniformly on any compact set in $\mathbb{R}^N$. Therefore if g satisfies

$$\liminf_{|x|\to\infty} |x|^{-\frac{2}{2-\alpha}} \ln \tilde{g}(|x|) > N^{\frac{1}{2-\alpha}}, \tag{3.28}$$

then there holds

$$J_n(t_n,x) \to \infty \Longrightarrow \lim_{t_n\to 0} \underline{u}_{g_n}(t_n,x) = \infty, \tag{3.29}$$
$n\to\infty$

uniformly on compact subsets of $\mathbb{R}^N$. We fix $m>0$, denote by λ_m the first eigenvalue of $-\Delta$ in $H_0^1(B_m)$, with corresponding eigenfunction ϕ_m normalized by $\sup_{B_m}\phi_m = 1$ and set, for $\epsilon>0$,

$$W_{m,\epsilon}(t,x) = e^{-(t+\epsilon)\lambda_m}\Phi_\infty(t+\epsilon)\phi_m(x) \qquad \forall (t,x)\in Q_\infty^{B_m}.$$

Then

$$\begin{aligned}\partial_t W_{m,\epsilon} - \Delta W_{m,\epsilon} + W_{m,\epsilon}\ln^\alpha(W_{m,\epsilon}+1) &= W_{m,\epsilon}\Big(\ln^\alpha(W_{m,\epsilon}+1) - \ln^\alpha(\Phi_\infty(t+\epsilon)+1)\Big)\\ &\le 0.\end{aligned}$$

Since $\underline{u}_{g_n}$ increases to the prospective minimal solution $\underline{u}_{\tilde g}$, it follows due to (3.29) that there exists n_ϵ such that

$$\underline{u}_{\tilde g}(t_{n_\epsilon},x) \ge \underline{u}_{g_{n_\epsilon}}(t_{n_\epsilon},x) \ge W_{m,\epsilon}(t_{n_\epsilon}+\epsilon,x) \qquad \forall x\in B_m.$$

Last inequality in virtue of comparison principle implies

$$\underline{u}_g(t,x) \ge W_{m,\epsilon}(t+\epsilon,x) \qquad \forall (t,x)\in Q_\infty^{B_m}, t\ge t_{n_\epsilon}.$$

Letting $\epsilon\to 0$ yields $\underline{u}_g \ge W_{m,0}$ in $Q_\infty^{B_m}$. Since $\lim_{m\to\infty}\phi_m(x) = 1$, uniformly on any compact subset of $\mathbb{R}^N$ and $\lim_{m\to\infty}\lambda_m = 0$ we derive $\underline{u}_{\tilde g} \geqslant \Phi_\infty$ and finally $\underline{u}_g \geqslant \Phi_\infty$. This inequality together with (3.14) leads to $\underline{u} = \Phi_\infty$. □

Remark. In the case $\alpha = 2$, there holds

$$\int_0^t \ln^2(\omega_n(s)+1)ds \le 4\gamma(r_n)\int_0^{t\gamma(r_n)} (2+\tau)^{-2}d\tau \le t\gamma(r_n). \tag{3.30}$$

Therefore (3.25) is replaced by

$$A_n(t,x) \ge \gamma(r_n) - t\gamma^2(r_n) - \frac{N(r_n+|x|)^2}{4t} - N\ln(r_n+|x|) - \frac{N}{2}\ln t := B_n(t,x). \tag{3.31}$$

A similarly, there exists $t_n>0$ where $t\mapsto B_n(t,x)$ is maximum and in that case

$$\begin{aligned}B_n(t_n,x) = \gamma(r_n) - N\ln(r_n+|x|) - \frac{N+\sqrt{N^2+4N(r_n+|x|)^2\gamma^2(r_n)}}{4}\\ - \frac{N(r_n+|x|)^2\gamma^2(r_n)}{N+\sqrt{N^2+4N(r_n+|x|)^2\gamma^2(r_n)}} - \frac{N}{2}\ln\left(\frac{N(r_n+|x|)^2}{N+\sqrt{N^2+4N(r_n+|x|)^2\gamma^2(r_n)}}\right),\end{aligned}$$

which yields

$$B_n(t_n,x) = \gamma(r_n) - r_n\gamma(r_n)(N^{\frac{1}{2}} - \nu_n(x)), \tag{3.32}$$

where $\nu_n(x)\to 0$ as $n\to\infty$ uniformly on any compact set in $\mathbb{R}^N$. Thus $B_n(t_n,x)\to -\infty$ as $n\to\infty$. A similar type of computation shows that the expression $I_n(t,x)$ defined in (3.17) converges to 0, whatever is the sequence $\{r_n\}$ converging to ∞.

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